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TRIANGULAR (0, 1)-MATRICES WITH PRESCRIBED ROW AND COLUMN SUMS

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This paper considers the existence of triangular matrices with specified row and column sums. Let $R=(r_1, r_2, \ldots, r_m)$ and $S=(s_1, s_2, \ldots, s_n)$ be vectors with nonnegative integral entries. Then $\mathfrak{A}(R, S)$ is the set of $m\times n$ (0, 1)-matrices with ith row sum r_i and jth column sum s_j. Let $m=n$ and let $\mathfrak{A}(R, S)$ be nonempty with $r_1\geq r_2\geq\cdots\geq r_n$, $s_1\geq s_2\geq\cdots\geq s_n$. Then there exists a matrix in $\mathfrak{A}(R, S)$ with a triangular block of 0's if and only if $r_i\leq n-i+1$ and $s_i\leq n-i+1$ for $1\leq i\leq n$. These are the simplest conditions one could hope for and are a substantial simplification of network flow conditions that can be obtained.

Using our result one can obtain that there is a matrix $A\in\mathfrak{A}(R, S)$ with $\operatorname{per}(A)=1$ if and only if there is a matrix $A\in\mathfrak{A}(R, S)$ with $A\geq P$ for some permutation matrix P (easy to check) and the above inequalities hold.

1. Introduction

The class $\mathfrak{A}(R, S)$ consists of all (0, 1)-matrices with given row and column sums. Let $R=(r_1, r_2, \ldots, r_m)$ and $S=(s_1, s_2, \ldots, s_n)$ be vectors with positive integral entries. We require $r_1+r_2+\cdots+r_m=s_1+s_2+\cdots+s_n$. We define the class $\mathfrak{A}(R, S)$ to be the set of $m\times n$-matrices with ith row sum r_i and jth column sum s_j. Simple necessary and sufficient conditions for the existence of matrices in $\mathfrak{A}(R, S)$ were found by Gale [9] and Ryser [12]. Theorem 2.2 specialized to $P=0$ yields these conditions. Much further research has been done. Brualdi has recently written an excellent survey article [4].

For most of our purposes, $m=n$ and we will call $\mathfrak{A}(R, S)$ of *order* n. We define R to be *monotone* if $r_1\geq r_2\geq\cdots\geq r_n$. The same definition applies to S.

In this paper we determine necessary and sufficient conditions for there to exist a triangular matrix in $\mathfrak{A}(R, S)$. Section 2 discusses the general class of problems of the form: does there exist a matrix $A\in\mathfrak{A}(R, S)$ with $A\leq B$ for a given matrix B? Such problems can be studied as a network flow problem and have a solution. Unfortunately, the conditions so obtained are too hard to apply. We will provide much simpler conditions in our special case.

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Section 3 provides some needed lemmas. In Section 4, we prove the main result. The necessary and sufficient conditions for the existence of a triangular matrix in $\mathfrak{A}(R, S)$ are as simple as you could hope for. The proof provides a simple inductive construction of such matrices. Manipulations of various inequalities verify that the construction works. Section 5 concludes with two applications of this result. The more interesting theorem gives simple necessary and sufficient conditions for the existence of a matrix $A \in \mathfrak{A}(R, S)$ with $\operatorname{per}(A) = 1$.

Most of these results have appeared in [2].

2. General existence theorem

A number of problems concerning $\mathfrak{A}(R, S)$ can be viewed as network flow problems. The following theorem was obtained by Mirsky [11]. It can also be derived from the integral supply-demand network flow theorem [6]. Fulkerson's result on the maximum number of disjoint permutation matrices covered by a given matrix is a fairly direct corollary [7].

Theorem 2.1. *There exists a matrix $A \in \mathfrak{A}(R, S)$ with $A \leq B$ if and only if for every submatrix B' of B, in rows given by the set I and columns given by J, we have*

$$N_1(B') \geq \sum_{i \in I} r_i - \sum_{j \in J} s_j, \tag{2.1}$$

where $N_1(B')$ denotes the number of 1's *in B'.*

In the general case, one would apply a network flow algorithm rather than testing (2.1) in the numerous cases. Similar sorts of questions can be answered by these techniques. Does there exist a matrix $A \in \mathfrak{A}(R, S)$ with $A + B \leq J$ where J is the matrix of 1's? One merely checks whether there is a matrix $A \in \mathfrak{A}(R, S)$ with $A \leq C$ where $C = J - B$. Let T be the (0, 1)-matrix of order n with a 1 in position (i, j) if and only if $i \leq n - j + 1$. The existence of a matrix $A \in \mathfrak{A}(R, S)$ with A triangular can be roughly rephrased as the existence of an $A \in \mathfrak{A}(R, S)$ with $A \leq T$. Our main theorem in Section 4 gives much simpler conditions than those given by Theorem 2.1. We provide Theorem 2.1 for comparison purposes.

We will be making use of the following result concerning $\mathfrak{A}(R, S)$. Let P be an $m \times n$ (0, 1)-matrix with column sums at most 1 and with no row or column sum exceeding the corresponding row or column sum in $\mathfrak{A}(R, S)$. we wish to know when there exists an $A \in \mathfrak{A}(R, S)$ with $A \geq P$. Let A^* be the $m \times n$ (0, 1)-matrix with ith row sum r_i and 1's wherever P has 1's. The remaining 1's are as far to the left as possible. Let the jth column sum of A^* be s_j^* and call the sequence (s_i^*) the *P-required conjugate* of the sequence (r_i). The following result appears in [1].

Theorem 2.2. *There exists a matrix $A \in \mathfrak{A}(R, S)$ with $A \geq P$ if and only if*

$$\sum_{i=1}^{t} s_i^* \geq \sum_{i=1}^{t} s_i \quad (1 \leq t \leq n), \tag{2.2}$$

where $s_1 \geq s_2 \geq \cdots \geq s_n$ and the (s_i^*) is the P-required conjugate of the sequence (r_i).

This is a substantial improvement over the network flows results that can be obtained and generalizes a result of Fulkerson on matrices with zero trace proven using network flows [8]. This also generalizes the existence theorem of Gale [9] and Ryser [12], which can be obtained by taking $P = 0$.

3. Some useful lemmas

This section proves three lemmas useful in Section 4. Our first result concerns the structure matrix as defined by Ryser [13]. Consider a class $\mathfrak{A}(R, S)$ and decompose any $A \in \mathfrak{A}(R, S)$ into blocks as follows

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}, \tag{3.1}$$

where W is of size $e \times f$. Let $N_i(M)$ denote the number of i's in matrix M. It turns out that $N_0(W) + N_1(Z)$ depends only on e, f and not on the choice of A. We define

$$t_{ef} = ef + r_{e+1} + r_{e+2} + \cdots + r_m - (s_1 + s_2 + \cdots + s_f). \tag{3.2}$$

We note that

$$N_0(W) + N_1(Z) = t_{ef}, \tag{3.3}$$

verifying our claim that this quantity is independent of the choice of A. The entries t_{ef} form what is called the structure matrix.

In Theorem 2.2, we may set $P = 0$. In that case denote A^* by $\bar{A}$ and let the sequence $(\bar{s}_i)$ be the 0-required conjugate of the sequence (r_i). The following result was noted by Brualdi [4] and the author [2].

Lemma 3.1. *The following special case holds*

$$t_{\bar{s}_f f} = \sum_{i=1}^{f} \bar{s}_i - \sum_{i=1}^{f} s_1. \tag{3.4}$$

Proof. Note that the right hand side of (3.4) being positive for each f yields that $\mathfrak{A}(R, S)$ is nonempty using Theorem 2.2. We may read off from $\bar{A}$ that $\bar{s}_f f$ is the number of 1's in the first f columns and $\bar{s}_f$ rows of $\bar{A}$. Also $r_{e+1} + r_{e+2} + \cdots + r_m$ is the number of 1's in the first f columns and the remaining rows of $\bar{A}$. Thus $\bar{s}_f f + r_{e+1} + r_{e+2} + \cdots + r_m$ is equal to $\bar{s}_1 + \bar{s}_2 + \cdots + \bar{s}_f$. Thus (3.4) follows from the definition (3.2).

Lemma 3.2. *Consider the decomposition of $A \in \mathfrak{A}(R, S)$ as in (3.1) where W is of size $\bar{s}_f \times f$. Then*

$$N_1(X) - N_0(W) = \sum_{i=f+1}^{n} \bar{s}_i. \tag{3.5}$$

Proof. This holds for any $\mathfrak{A}(R, S')$. A matrix $A \in \mathfrak{A}(R, S)$ can be obtained from $\bar{A}$ by shifting 1's to the right. The equation (3.5) is true for $\bar{A}$. Shifting a 1 from W to X increases both $N_1(X)$ and $N_0(W)$ and so $N_1(X) - N_0(W)$ remains constant which proves the lemma.

We recall the definition of T as the (0, 1)-matrix of order n with a 1 in position (i, j) if and only if $i \leq n - j + 1$.

Lemma 3.3. *Consider $\mathfrak{A}(R, S)$ with $m = n$ and R, S monotone. Then there exists a matrix $A \in \mathfrak{A}(R, S)$ with $A \leq PTQ$ where P and Q are permutation matrices if and only if there exists a matrix $C \in \mathfrak{A}(R, S)$ with $C \leq T$.*

Proof. Consider a matrix $A \in \mathfrak{A}(R, S)$ with two rows k and l as follows. We have $k < l$ with row l having 0,s in columns $j_1, j_2, \ldots, j_r$ and row k having 0's in columns $j_1, j_2, \ldots, j_r, j_{r+1}, \ldots, j_s$. Since R is monotone, we have $r_k \geq r_l$ and so certainly some 0's are unaccounted for. We claim that there is a matrix $B \in \mathfrak{A}(R, S)$ with row k having 0's in columns $j_1, j_2, \ldots, j_r$ and row l having 0's in columns $j_1, j_2, \ldots, j_s$ and the remaining rows are the same as those in A.

Consider a column j_t where $r < t \leq s$. Possibly row l has a 0 in column j_t already. If row l has a 1 in column j_t then, since $r_k \geq r_l$, there is a column i_t ($i_t \notin \{j_1, j_2, \ldots, j_s\}$) with a 1 in row k and a 0 in row l. Consider the following two matrices

$$\text{(i)} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \text{(ii)} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.6}$$

Replacing a submatrix of a matrix A equal to (i) by (ii) or vice versa does not affect the row or column sums. Either replacement is called an *interchange*. In our case we have an interchange in A involving rows k and l and columns j_t and i_t that puts a 0 in row l in column j_t. After a series of such interchanges, which only involve rows k and l, we obtain a matrix B as claimed.

The same argument works for columns. Now suppose $A \leq PTQ$ where P and Q are permutation matrices. Thus A has 0's whenever PTQ has 0's. After a series of the above operations, we may obtain a matrix $C \in \mathfrak{A}(R, S)$ with C having 0's where T has 0's. Thus $C \leq T$ as desired. The reverse implication is easy.

The same arguments of this lemma would apply not just to T but to any matrix $T' \in \mathfrak{A}(R', S')$ where $|\mathfrak{A}(R', S')| = 1$. Haber used this idea when looking for matrices in $\mathfrak{A}(R, S)$ with a large block of 0's while considering the minimum term rank of all matrices in $\mathfrak{A}(R, S)$ [10].

4. Triangular matrix theorem

We now tackle our main result.

Theorem 4.1. *Let $\mathfrak{A}(R, S)$ be a nonempty class of order n with R, S monotone. Then there exists an $A \in \mathfrak{A}(R, S)$ with*

$$A = \begin{bmatrix} * & \diagup * \\ * \diagup & 0 \end{bmatrix} \tag{4.1}$$

where the $*$'s *denote arbitrary entries if and only if*

$$r_i \leqslant n-i+1 \quad \text{and} \quad s_i \leqslant n-i+1 \qquad \text{for } 1 \leqslant i \leqslant n. \tag{4.2}$$

Proof. Assume such a matrix A exists. Thus the inequalities (4.2) must hold.

Assume the inequalities hold. We will prove the existence of a matrix A satisfying (4.1) using induction on n. The arguments yield a simple inductive construction. The matrix A exists trivially for $n = 1$. Assume the theorem is true for $n-1$ and we will prove it for n. Assume the inequalities (4.2) hold. We will consider a possible first row for A. Let L be the (0, 1)-matrix of order n with a 1 in column i and row 1 for each i where $s_i = n-i+1$ and 0's elsewhere. Let M be the (0, 1)-matrix of order n with r_1 1's in the first row, 0's elsewhere with the 1's as far to the left as possible subject to the condition $M \geqslant L$. Similar interchange arguments to those used in Lemma 3.3 verify that there is a matrix $A \in \mathfrak{A}(R, S)$ with $A \geqslant L$ if and only if there is a matrix $B \in \mathfrak{A}(R, S)$ with $B \geqslant M$.

We define a (0, 1)-matrix N in a similar way to M. Let N have r_1 1's in the first row and 0's elsewhere. We require that $N \geqslant L$ and that the remaining 1's are placed in the remaining columns with largest column sums. In the event that there are p 1's to be placed in the q columns with the next largest column sum, then these 1's are placed as far to the right as possible. For example, with $S = (5, 5, 4, 4, 4, 3, 1, 1)$ and $r_1 = 6$ we define the first row of N as $(1, 1, 0, 1, 1, 1, 0, 1)$. A column permutation is sufficient to verify that there is a matrix $B \in \mathfrak{A}(R, S)$ with $B \geqslant M$ if and only if there is a matrix $C \in \mathfrak{A}(R, S)$ with $C \geqslant N$.

If such a matrix C exists, then the first row of C is the first row of N. Let D be the submatrix of C obtained by deleting the first row and last column and let $D \in \mathfrak{A}(R', S')$. We have chosen N so that R' and S' are monotone and satisfy the inequalities (4.2) where n is replaced by $n-1$. Certainly $\mathfrak{A}(R', S')$ is nonempty so by induction there is a matrix $E \in \mathfrak{A}(R', S')$ in triangular form, i.e.

$$E = \begin{bmatrix} * & \diagup * \\ * \diagup & 0 \end{bmatrix}. \tag{4.3}$$

Let F be the (0, 1)-matrix of order n obtained from E by adding a first row of 0's and a last column of 0's. Then $F+N \in \mathfrak{A}(R, S)$ and $F+N$ is the desired matrix in (4.1) with $P=Q=I$.

To complete the proof, we must show that there is a matrix $A \in \mathfrak{A}(R, S)$ with $A \geqslant L$ which is an application of Theorem 2.2. We must first show that L satisfies the hypotheses. By construction, L has column sums at most 1 and equal to 1 only when $s_i = n-i+1 \geqslant 1$. Let L have c 1's. Then we must show that $r_1 \geqslant c$.

Form the (0, 1)-matrix of order n with jth column sum s_j and 1's as far to the top as possible. It will look like

$$\begin{bmatrix}
 & \text{1's} & & & & & 1\\
 & & & & & 1 \cdot & \\
 & & & * & 0 & & \\
 & & & 0 \cdot & & & \\
 & & & i & & & \\
 & 1 \cdot & & & & & \\
* & 0 & & & & & \\
 \cdot & & & & & \text{0's} & \\
0 & & & & & &
\end{bmatrix}, \tag{4.4}$$

with c 1's on the reverse diagonal. The entries marked $*$ are unspecified. We consider the case that the matrix in (4.4) has as few 1's as possible subject to the monotonicity of S and the inequalities (4.2). Having determined this, it will imply a minimum value for r_1 subject to the monotonicity of R and the inequalities (4.2). Thus we set the entries marked $*$ to zero. Now it can be seen that the c 1's on the reverse diagonal can be shifted, one at a time, as far to the upper right as possible without increasing the number of 1's. We obtain the matrix

$$\begin{bmatrix}
 & \text{1's} & & & 1\\
 & & & 1 \cdot & \\
 & & 1 & & \\
 & & 0 & & \\
 \text{0's} & \cdot & & & \\
 0 \cdot & & & \text{0's} & \\
0 & & & &
\end{bmatrix}. \tag{4.5}$$

This matrix has the fewest number of 1's possible among all matrices with monotone column sums with c 1's on the reverse diagonal and satisfying the inequalities (4.2). By considering the transpose of (4.5) and recalling that the row sums are monotone and satisfy the inequalities (4.2), we deduce that $r_1 \geqslant c$. Thus L satisfies the hypotheses of Theorem 2.2.

Let the sequence (s_i^*) be the L-required conjugate of the sequence (r_i). By

Theorem 2.2, we must show

$$\sum_{i=1}^{t} s_i^* \geq \sum_{i=1}^{t} s_i, \tag{4.6}$$

for $1 \leq t \leq n$ for there to exist an $A \in \mathfrak{A}(R, S)$ with $A \geq L$. We will verify (4.6) for a given value of t. Let k be the number of i for which $s_i = n - i + 1$ and $i > t$. As before let A^* be denoted by $\bar{A}$ when $P = 0$ in Theorem 2.2 and let the sequence $(\bar{s}_i)$ be the 0-required conjugate of the sequence (r_i). We may obtain

$$\sum_{i=1}^{t} \bar{s}_i - \sum_{i=1}^{t} s_i^* = \begin{cases} k & \text{for } \ r_1 < t, \\ \max(0, k - (r_1 - t)) & \text{for } \ r_1 \geq t. \end{cases} \tag{4.7}$$

The number on the right corresponds to the number of 1's that have to be shifted from the first t columns of $\bar{A}$ to the last $n - t$ columns to form the matrix A^* in Theorem 2.2 for $P = L$.

We will obtain (4.6) from (4.7). Decompose any $A \in \mathfrak{A}(R, S)$ into blocks as follows

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}, \tag{4.8}$$

where W is of size $\bar{s}_t \times t$ (possibly $\bar{s}_t = 0$).

Lemma 3.1 tells us that

$$\sum_{i=1}^{t} \bar{s}_i - \sum_{i=1}^{t} s_i = N_0(W) + N_1(Z). \tag{4.9}$$

We claim that the expression in (4.9) is as large as that in (4.7). In that case (4.6) would be true, finishing our proof.

Case 1: $r_1 < t$.

Thus $\bar{s}_t = 0$ and $N_0(W) = 0$. Also $N_1(Z) = s_{t+1} + s_{t+2} = \cdots + s_n$ and so $N_0(W) + N_1(Z) \geq k$ as desired.

Case 2: $r_1 \geq t$.

Let $r_1 = t + u$ with $u \geq 0$. We may assume $u < k$, otherwise the expression in (4.7) is zero verifying our claim easily. There are k columns i for which $s_i = n - i + 1$ and $i > t$. Using the monotonicity of column sums, we deduce that

$$N_1(X) + N_1(Z) \geq (n - k - t)k + \tfrac{1}{2}k(k + 1), \tag{4.10}$$

using the same arguments that were used on L to verify that $r_1 \geq c$. Lemma 3.2 tells us

$$N_1(X) - N_0(W) = \sum_{i=t+1}^{n} \bar{s}_i. \tag{4.11}$$

We may maximize this by taking $\bar{s}_t = n - t + 1$ and all the r_i's as large as possible for $1 \leq i \leq n - t + 1$, subject to the monotonicity of R and the inequalities of (4.2). Thus

$$N_1(X) - N_0(W) \leq (n - t + 1)u - \tfrac{1}{2}u(u + 1). \tag{4.12}$$

Combining this with (4.10), we obtain

$$N_0(W)+N_1(Z)\geqslant (k-u)(n-t+1-\tfrac{1}{2}(k+u+1)). \tag{4.13}$$

Since $\frac{1}{2}(k+u+1)\leqslant k$ and $n\geqslant t+k$, we have

$$N_0(W)+N_1(Z)\geqslant k-u. \tag{4.14}$$

Having defined $r_1=t+u$, we find that $k-u$ is the expression in (4.7) and thus our claim is verified in this case as well.

It is important to note that the matrix we constructed has $A\leqslant T$ as in (4.1). Consider $R=S=(0,1)$. The inequalities (4.2) are true here and $\mathfrak{A}(R,S)$ is nonempty. However R and S are not monotone and there does not exist a matrix in $\mathfrak{A}(R,S)$ with a 0 in position (2,2). There is nonetheless a matrix in $\mathfrak{A}(R,S)$ in triangular form.

5. Applications

Theorem 5.1. *Let $\mathfrak{A}(R,S)$ be a nonempty class of size $m\times n$ with R and S monotone. Then there exists an $A\in\mathfrak{A}(R,S)$ with*

$$PAQ=\begin{bmatrix} * & * \\ * & 0 \end{bmatrix}, \tag{5.1}$$

when P and Q are permutation matrices and the triangular block of zeros is of size $m+n-q+1$ if and only if

$$\begin{aligned} r_i&\leqslant q-i+1 \quad (1\leqslant i\leqslant m),\\ s_j&\leqslant q-j+1 \quad (1\leqslant j\leqslant n). \end{aligned} \tag{5.2}$$

Proof. We define a class $\mathfrak{A}(R',S')$ from $\mathfrak{A}(R,S)$ as follows. Let $R'=(r_1',r_2',\ldots,r_q')$ with $r_i'=r_i$ for $1\leqslant i\leqslant m$ and $r_i'=0$ for $i>m$. Let $S'=(s_1',s_2',\ldots,s_q')$ with $s_j'=s_j$ for $1\leqslant j\leqslant n$ and $s_j'=0$ for $j>n$. A matrix $A\in\mathfrak{A}(R,S)$ can be extended to a matrix $B\in\mathfrak{A}(R',S')$ by adding rows and columns of 0's. Thus $\mathfrak{A}(R',S')$ is nonempty and R',S' are monotone.

Assume such an A exists. Extend to a matrix in $\mathfrak{A}(R',S')$ and apply Lemma 3.3 to verify that the inequalities (5.2) hold.

Assume the inequalities (5.2) hold. Then the inequalities of Theorem 4.1 hold with $n=q$ in $\mathfrak{A}(R',S')$. Thus there exists a matrix in triangular form in $\mathfrak{A}(R',S')$ which, after stripping off the rows and columns of zeros, yields the desired matrix A with $P=Q=I$.

The above theorem is equivalent to Theorem 4.1 and extends its applicability. We now consider classes $\mathfrak{A}(R,S)$ of order n matrices which contain a matrix with permanent equal to 1. For such matrices to exist, there must exist a matrix

$A\in\mathfrak{A}(R,S)$ with $A\geqslant P$ for some permutation matrix P. This is an easy condition to verify using a theorem of Brualdi and Ross [5]. Let $\mathbf{1}=(1,1,\ldots,1)$ be the vector of n 1's. Then there is a matrix $A\in\mathfrak{A}(R,S)$ with $A\geqslant P$ for some permutation matrix P if and only if $\mathfrak{A}(R-\mathbf{1},S-\mathbf{1})$ is nonempty.

Theorem 5.2. *Let $\mathfrak{A}(R,S)$ be a nonempty class of order n with R, S monotone. Then there exists an $A\in\mathfrak{A}(R,S)$ with* $\operatorname{per}(A)=1$ *if and only if $\mathfrak{A}(R-\mathbf{1},S-\mathbf{1})$ is nonempty and*

$$r_i\leqslant n-i+1,\quad s_i\leqslant n-i+1\qquad(1\leqslant i\leqslant n).\tag{5.3}$$

Proof. Let $A\in\mathfrak{A}(R,S)$ with $\operatorname{per}(A)=1$. Since $A\geqslant P$ for some permutation matrix P we obtain that $\mathfrak{A}(R-\mathbf{1},S-\mathbf{1})$ is nonempty. Using a theorem of Brualdi [3], we have that for some permutation matrices P, Q,

$$PAQ=\begin{bmatrix} * & 1\\ 1 & 0\end{bmatrix}.\tag{5.4}$$

Thus using Lemma 3.3, we obtain the inequalities (5.3).

Assume that $\mathfrak{A}(R-\mathbf{1},S-\mathbf{1})$ is nonempty and (5.3) holds. Let $R-\mathbf{1}=(r_1',r_2',\ldots,r_n')$ and $S-\mathbf{1}=(s_1',s_2',\ldots,s_n')$. Using (5.3), we obtain

$$r_i'\leqslant(n-1)-i+1,\quad s_i'\leqslant(n-1)-i+1\qquad(1\leqslant i\leqslant n-1).\tag{5.5}$$

Certainly $R-\mathbf{1},S-\mathbf{1}$ are monotone and so we apply Theorem 5.1 to obtain a matrix $B\in\mathfrak{A}(R-\mathbf{1},S-\mathbf{1})$ of the form

$$B=\begin{bmatrix} * & 0\\ 0 & 0\end{bmatrix}.\tag{5.6}$$

Let $A=B+M$ where M is the matrix with 1's in positions $(i,n-i+1)$ for $1\leqslant i\leqslant n$ and 0's elsewhere. Then $A\in\mathfrak{A}(R,S)$ and $\operatorname{per}(A)=1$ as desired.

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